

SIMPLE CURVES ON SURFACES

IGOR RIVIN

To Larry Siebenmann on the occasion of his sixtieth birthday

ABSTRACT. We study simple closed geodesics on a hyperbolic surface of genus g with b geodesic boundary components and c cusps. We show that the number of such geodesics of length at most L is of order $L^{6g+2b+2c-6}$. This answers a long-standing open question.

Let \mathcal{S} be a hyperbolic surface of genus g with c cusps and b boundary components. In this paper we study the set of simple (that is, without self-intersections) closed geodesics on \mathcal{S} . More precisely we study the counting function $\mathcal{N}(L, \mathcal{S})$ – the number of simple geodesics of length no greater than L on the surface \mathcal{S} . We show that there are constants c_1 and c_2 (depending only on \mathcal{S}), such that

$$(1) \quad c_1 L^{6g-6+2b+2c} \leq \mathcal{N}(L, \mathcal{S}) \leq c_2 L^{6g-6+2b+2c}.$$

The estimate (1) should be put into proper perspective, and it is with this end that we give the following historical summary on the study of simple closed curves on surfaces. This study goes back all the way to the beginning of the subject of geometry and topology of surfaces (that is, the work of Henri Poincaré and Max Dehn), and some of the subsequent work we will mention is a developement (if not actually a repetition) of this work. Being as it may, one line of inquiry has been group theoretic: suppose γ is an element in the fundamental group of S , how do we decide whether or not γ is represented by a simple loop? This question was probably known to Nielsen for the case of a punctured torus, however, the earliest reference known to me is the paper of Osborne and Zieschang [14]. In the general case, the first reasonable algorithm for determining whether an element of $\pi_1(S)$ can be represented by a simple curve was given by H. Zieschang [20, 21], and D. Chillingworth [5, 6], following earlier work of Reinhart [16] – Zieschang’s algorithm is primarily group-theoretic, whilst Reinhart–Chillingworth is more geometric. This work has been rendered more explicit by Birman and Series [2, 4], and roughly at the same time Cohen and Lustig [7, 10] had extended the Birman–Series algorithm to determine the minimum number of intersections

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between curves representing two homotopy classes (which includes the self-intersection number of a single curve as a special case).

Unfortunately, while this algorithmic work is very interesting (the words in the fundamental group represented by simple closed curves are a direct generalisation of Sturm sequences – these are precisely the words in the free group on two generators represented by simple curves on the punctured torus, as remarked by Birman and Series in their papers), they do not seem to be usable for estimating the number of such words as a function of the length of the word (which is, up to constant factors, the same as the length of the corresponding geodesic).

This brings us to the counting question. It seems that the problem of counting *all* geodesics of bounded length has been resolved almost entirely, due to the work of Delsarte, Huber, and Selberg in the constant curvature case, and Margulis, Bowen, Ruelle, and others in the variable curvature case. In all cases the estimate is that the number of geodesics of length bounded above by L is *asymptotic* to $\exp(hL)/L$, where h is the topological entropy of the geodesic flow. In particular, $h = 1$ for every finite area hyperbolic surface. Thus, the growth of the number of all geodesics on all such surface depends (up to first order) neither on the topology, nor on the actual hyperbolic metric. This would seem to indicate that the set of all closed geodesics is not a very geometric object. Now, for simple geodesics, things are a lot more subtle. For example, for the simplest hyperbolic surface – the thrice-punctured sphere – there are none. After that, things become more complicated. For the four times punctured sphere, Beardon, Lehner, and Sheingorn [1] had shown that the number of simple geodesics grew at least linearly and at most quadratically, as a function of length. Since the four-punctured sphere and the once-punctured torus are essentially the same, this implies the same estimate for the torus. On the other hand, Thurston's theories of measured foliations (see [8]) and (dually) projective laminations imply that:

$$(2) \quad c_1 L^{6g-6+2b+2c} \leq \mathcal{M}(L, \mathcal{S}) \leq c_2 L^{6g-6+2b+2c},$$

where $\mathcal{M}(L, \mathcal{S})$ is The number of collections of pairwise non-intersecting simple closed geodesics of total length no greater than L on \mathcal{S} .

It *is* allowed to take multiple copies of any given curve; its contribution to the total length is then multiplied by the multiplicity.

Notation. Such a collection of curves will henceforth be called a multicurve.

Since on a once-punctured torus every multicurve is connected (though possibly covers itself multiple times), the estimate (1) for the torus follows, in essence, from the estimate (2) when $g = 1$, and $b + c = 1$. This also implies the estimate (1) for the 4-times punctured sphere. This estimate only appeared in print in my paper with Greg McShane [11]. In that paper (see also [12]) we actually show a much stronger result: $N(L, S)$ for S a

punctured torus is *asymptotic* to $c_S L^2$, where the coefficient c_S depends on the hyperbolic structure, varies real-analytically over moduli space of tori, and goes to infinity at infinity of moduli space (hence is not constant).

For S a surface of genus 2, the estimate (1) follows from the work of Haas and Susskind [9].

In general, Birman and Series [3] have shown that for any genus, the number of simple curves (actually the number of curves with a bounded number of self-intersections) grows at most polynomially, with the exponent depending on the topological type of the surface. For simple curves their result follows immediately from the estimate (2), which provides an upper bound for the number of simple closed curves of bounded length. The harder part (and the subject of this paper) is proving the lower bound. It should be noted that this is claimed (indirectly) in the paper [15]. However, the argument there is extremely incomplete, and has never been generally accepted.

More recently, Geoff Mess (private communication) has claimed to have improved the estimate (2) for $M(S, L)$ (the number of *multicurves* to an asymptotic result (crudely speaking, showing that one can choose c_1 and c_2 arbitrarily close to each other). Furthermore, he has claimed to be able to show analytic variation of the resulting constant over moduli space.

Of course, a really interesting question is whether there is an asymptotic form of (1). The argument proving the estimate (1) (and occupying the rest of this paper) seems to indicate that such a result should exist, but it seems difficult.

Here is an outline of the rest of the paper:

Section 1 contains some background facts. In Sections 2 and 3 the basic method is developed and used to prove estimate (1) for $g = 0, 1$. The case of arbitrary genus requires a couple of other refinements, and is addressed in Section 4.

Notation. In the sequel, whenever constants are used (denoted by c , c_1 , etc), it is to be understood that these depend solely on the hyperbolic metric on the surface in question. The same letter can (and does) denote different numbers in different places in the paper.

1. BACKGROUND

In this section we assemble some necessary background facts.

Theorem 1.1. *Let S be a hyperbolic surface, and \mathcal{C} a cusp of S . Then there is an embedded horodisk neighborhood of \mathcal{C} of area 2 which contains no simple closed geodesic of S .*

Proof. This theorem goes back to Poincaré, for a proof see [11]. □

Theorem 1.2. *Let γ , β be simple closed geodesics on a hyperbolic surface S , and let $\mathcal{T}_\beta(\gamma)$ be the Dehn twist of γ around β . Then $\ell(\mathcal{T}_\beta(\gamma)) \leq \ell(\gamma) + i(\gamma, \beta)\ell(\beta)$, where $i(\gamma, \beta)$ is the geometric intersection number of γ and β .*

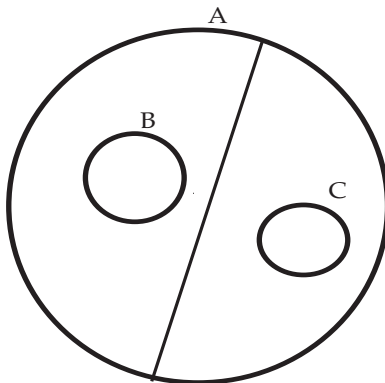


FIGURE 1. first kind

Proof. This follows (among other things) from the variational formula of Wolpert [19] on the change of length of curves under earthquake deformations. See also [18]. \square

Theorem 1.3. *Given two closed curves β_0, γ_0 , the smallest geometric intersection number between curves β, γ , freely homotopic to β_0, γ_0 , respectively, is realized for β, γ geodesic.*

Proof. This was also known to Poincaré, for a proof see [17]. \square

From now on, $\Gamma_{A_1 \dots A_n}$ shall denote the sphere S^2 missing n disks (usually equipped with a hyperbolic metric with n geodesic boundary components).

Theorem 1.4. *There is only one homotopy class of non-boundary-parallel simple curves on Γ_{ABC} beginning and ending on the same boundary component A .*

Proof. See diagram 1. \square

Theorem 1.5. *There is only one homotopy class of curves on Γ_{ABC} joining boundary component A to boundary component B .*

Proof. See diagram 2. \square

Theorem 1.4 is relevant in view of:

Theorem 1.6. *No geodesic segment in Γ_{ABC} can be boundary parallel.*

Proof. There are no geodesic bigons in \mathbb{H}^2 . \square

These observations are sufficient for us to begin counting simple curves.

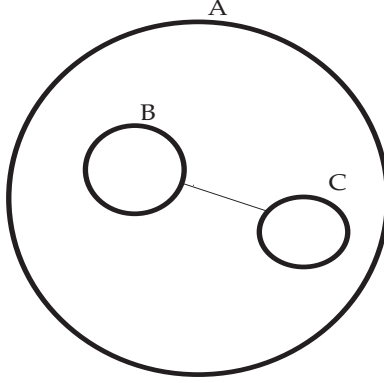


FIGURE 2. second kind

2. LOW GENUS

To be systematic, we start with the easiest case:

Theorem 2.1. *Any simple closed geodesic on Γ_{ABC} is a boundary component.*

Proof. Exercise. □

Theorem 2.2. *For any hyperbolic structure \mathcal{S} on the 4-punctured sphere, $\exists c_{\mathcal{S}}, L_0 > 0$, such that for any $L > L_0$, the number of simple closed geodesics on \mathcal{S} for length not exceeding L_0 is not less than $c_{\mathcal{S}}L^2$.*

Remark 2.3. *Of course, a stronger statement follows from [11, 12], see the Introduction, but we use the proof of this result to introduce the techniques and notation for the rest of the paper.*

Proof. Let Γ_{ABCD} be the 4-punctured sphere in question. Pick a simple loop E , separating Γ_{ABCD} into two thrice-punctured spheres, Γ_{ABE} , and Γ_{CDE} (see diagram 3). Let γ be a simple geodesic. There are a finite number of such which do not intersect the separating curve E (either 4 or 0, depending on whether or not one counts boundary curves). The curve γ could be homotopic to E , but if not, it must intersect E transversely in $2k$ points. Let γ be such a curve. Note that $\gamma \cap \Gamma_{ABE}$ consists of k geodesic segments, having all of their endpoints on E . Up to homotopy, there is only one way to thus place k segments in Γ_{ABE} , by Theorem 1.4. See Figure 4. The intersection of γ with Γ_{CDE} looks similar. Note that the length of $\gamma \cap \Gamma_{CDE} \asymp k$.

Consider the inverse operation: Given two diagrams which look like figure 4 we can glue them together with a *rational twist* $\frac{p}{2k}$. The integer part $t = \lfloor \frac{p}{2k} \rfloor$ corresponds to twisting γ t times around E . The fractional part correspond to the change of the identification map: A diagram (with an orientation) has a canonical labelling (shown in Figure 4). Gluing with no

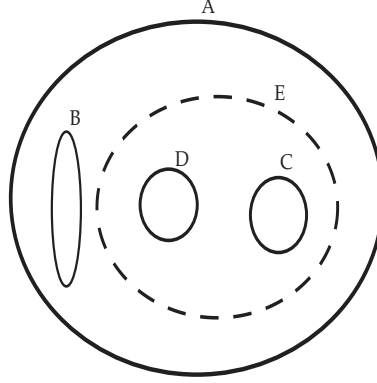
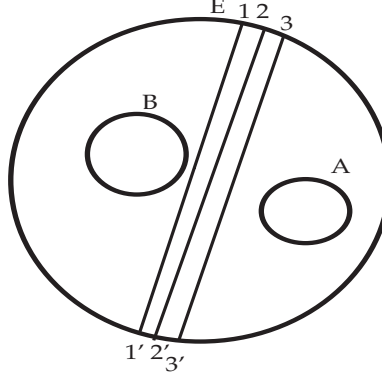


FIGURE 3. A four punctured sphere divided in two

FIGURE 4. k segments from E to E

twist corresponds to attaching the strand labelled 1 to one labelled 1, and so on. Twisting by q corresponds to gluing the strand labelled l to the strand labelled $l + q$, if $l + q < k$, to $(2k - (l + q))'$ if $l < k$, $l + q > k$ and so on.

It is not hard to see that the twist $\frac{p}{2k}$ leads to a *connected* curve if and only if p and $2k$ are relatively prime. By observation 1.2, the length of γ twisted by $\frac{p}{2k}$ is bounded above by $p\ell(E)/2$, while the number of twists not exceeding N leading to connected curves is

$$(3) \quad \frac{\phi(k)}{k} 2kN = 2\phi(k)N,$$

where ϕ denotes the Euler totient function.

By the previous observations, the length of curves obtained thereby is bounded above by $kN\ell(E)$, so to obtain curves of length not exceeding L , we must take $N \leq \frac{L}{k\ell(E)}$, thus, for a fixed k we have

$$\frac{\phi(k)L}{k\ell(E)}$$

curves. Since k could be anything up to cL (the constant c depending on the metric on Γ_{ABCD}), we see that

$$(4) \quad \mathcal{N}(L, \Gamma_{ABCD}) \geq \sum_{k=1}^{cL} \frac{L}{\ell(E)} \frac{\phi(k)}{k} \geq c' L^2,$$

where we use $\mathcal{N}(L, \mathcal{S})$ to denote the number of simple geodesics on a surface \mathcal{S} of length not exceeding L . The last inequality in Eq. (4) is the consequence of Lemma 2.4 below (the argument is standard in number theory; we give it here for completeness). \square

Lemma 2.4.

$$\sum_{m=1}^n \frac{\phi(m)}{m} = \frac{6}{\pi^2} n + O(\log n),$$

where ϕ denotes the Euler totient function.

Proof. First, note that

$$(5) \quad \sum_{d|n} \phi(d) = n.$$

This follows, for example, from the observation that the number of elements of order $d|n$ in the cyclic group of order n is equal to $\phi(d)$. From equation (5), we have, by Möbius inversion, that

$$(6) \quad \frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$

Using equation (6) we have

$$S(n) = \sum_{m \leq n} \frac{\phi(m)}{m} = \sum_{m \leq n} \sum_{d|m} \frac{\mu(d)}{d}.$$

Changing the order of summation, we see that

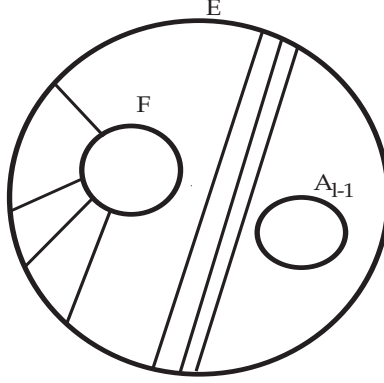
$$S(n) = \sum_{d \leq n} \sum_{j \leq \frac{n}{d}} \frac{\mu(d)}{d} = \sum_{d \leq n} \left\lfloor \frac{n}{d} \right\rfloor \frac{\mu(d)}{d},$$

where $\lfloor x \rfloor$ denotes the integer part of x . Since $|x - \lfloor x \rfloor| < 1$, we have the estimate

$$|S(n) - S_0(n)| < \sum_{d \leq n} \left| \frac{\mu(d)}{d} \right| < \sum_{d < n} \frac{1}{d} = \log(n) + O(1),$$

where

$$S_0(n) = \sum_{d \leq n} \frac{n}{d} \frac{\mu(d)}{d} = n \sum_{d \leq n} \frac{\mu(d)}{d^2}.$$

FIGURE 5. 4 segments from F to E , 3 segments from E to E

Note that

$$\sum_{j=1}^{\infty} \frac{\mu(j)}{j^s} = \frac{1}{\zeta(s)},$$

where ζ is the Riemann ζ function, while

$$\left| \sum_{j=n+1}^{\infty} \frac{\mu(j)}{j^2} \right| \leq \sum_{j=n+1}^{\infty} \frac{1}{j^2} = O(n^{-1}).$$

Putting all these estimates together, we get the conclusion of the lemma. \square

This analysis will now be extended to deal with a sphere with k boundary components. We will prove

Theorem 2.5. *The number of simple geodesics of length bounded by L on a sphere with c boundary components grows like L^{2c-6} .*

Proof. As before, take the sphere $\Gamma_{A_1 \dots A_k}$ and cut it into a sphere $\Gamma_{A_1 \dots A_{k-2} E}$ with $k-1$ boundary components and a pair of pants $\Gamma_{EA_{k-1} A_k}$. We will, again, count the simple geodesics γ which intersect E $2k$ times, then sum over the possible values of k . The intersection of such a curve with $\Gamma_{EA_{k-1} A_k}$ was already studied in the proof of Theorem 2.2. It remains to analyse $\gamma \cap \Gamma_{A_1 \dots A_{k-2} E}$. This is a collection of k disjoint segments having all of their endpoints on E , and we will analyse this inductively. We will prove the following

Lemma 2.6. *Let $G = \Gamma_{A_1 \dots A_c}$ be a sphere with c boundary components. The number of k component multicurves on G of length bounded above by L is bounded below by a constant times $k^{c-3} L^{c-3}$.*

Proof of lemma 2.6. Consider a sphere with c boundary components, $\Gamma_{A_1 \dots A_{l-1} E}$. We cut off a 3-punctured sphere $\Gamma_{FA_{l-1} E}$. The intersection of γ with this sphere is a collection of segments, $2m$ of which go from F to E , while $k-m$ go from E to itself (see diagram 5).

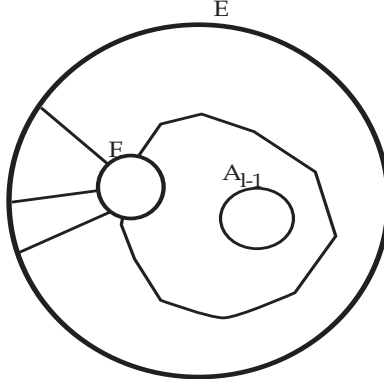


FIGURE 6. Another possibility

Remark 2.7. *There is another combinatorial possibility, shown in Figure 6, but as we are only interested in a lower bound, we ignore it.*

The intersection of γ with $\Gamma_{A_1 \dots A_{c-1} F}$ has $m < k$ connected components. Thus, we have the inequality

$$(7) \quad \mathcal{N}(\mathfrak{c}, k, L) \geq c \sum_{m=0}^k \int_0^L (L-x) d\mathcal{N}(\mathfrak{c}-1, m, x),$$

where the integral is in the sense of Stieltjes, and $\mathcal{N}(\mathfrak{c}, l, L)$ is the number of l -component multicurves of length bounded above by L on a sphere with \mathfrak{c} boundary components beginning and ending on a fixed component.

Remark 2.8. *$f(x) \asymp x^k, k \geq 0$, then*

$$(8) \quad \int_0^L (L-x) df(x) = \int_0^L f(x) dx \asymp x^{k+1},$$

integrating by parts.

Proof of inequality (7). Note that each term in the sum comes from the intersection of γ with $\Gamma_{A_1 \dots A_{c-1} F}$ having m components. In each case, if that intersection has length x , we can twist the length of that intersection is x , and we can twist a number of times around F to bring the length up to L . This number is proportional to $L-x$, (up to a constant, of order of length of F). \square

For example, for $c = 3$,

$$\mathcal{N}(3, l, L) = \begin{cases} 0 & L < L_0, \\ 1 & \text{otherwise.} \end{cases}$$

For $c = 4$,

$$(9) \quad \mathcal{N}(4, l, L) \geq clL,$$

by virtue of Remark 2.8. An easy inductive argument shows that

$$(10) \quad \mathcal{N}(\mathfrak{c}, l, L) \geq cl^{\mathfrak{c}-3} L^{\mathfrak{c}-3}.$$

□

To complete the proof of Theorem 2.5, use Lemma 2.6, and essentially repeat the counting argument in the proof of that Lemma *verbatim*. □

3. SURFACES OF GENUS ONE

To extend our estimates to surfaces of genus 1, we follow the same basic strategy: given a surface of genus 1 with c boundary components (we will denote such a surface by $T_{A_1 \dots A_c}$, we cut along a nonseparating simple curve E to obtain a surface of genus 0 with $c + 2$ boundary components: $\Gamma_{E_1 A_1 \dots A_c E_2}$. Given a simple curve γ on $T_{A_1 \dots A_c}$, $\gamma \cap \circ \Gamma_{E_1 A_1 \dots A_c E_2}$ is a collection of segments, each of whose endpoints lies either on E_1 or E_2 . Conversely, given such a collection of segments, of total length x , we know that by fractional twisting we can produce several curves $\gamma_1, \dots, \gamma_N$, by identifying the boundary components E_1 and E_2 with a fractional twist. The number N of such curves is of order $L - x$, just as in the proof of Theorem 2.2. We now need to count collections of segments as above, having k intersections with E_1 and l intersections with E_2 . In fact, for the purposes of induction we will count collections of segments having k intersections with E_1 and l intersections with E_2 , a quantity we shall denote by $\mathcal{N}(c, k, l, L)$ (L being the upper bound on the length). For $c = 1$, the integers k and l determine the curve completely, so

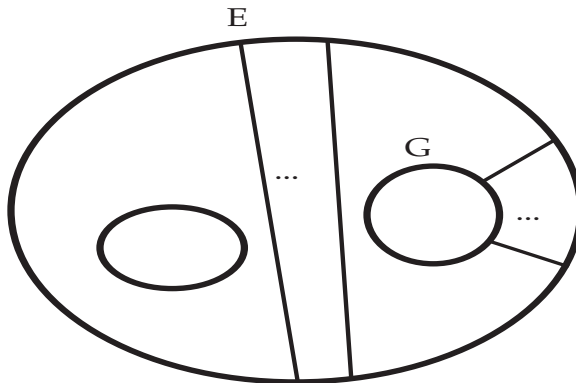
$$(11) \quad \mathcal{N}(3, k, L) = \begin{cases} 0 & L < L_0, \\ 1 & \text{otherwise.} \end{cases}$$

For bigger c , we cut the $\Gamma_{E_1 A_1 \dots A_c E_2}$ into two pieces: $\Gamma_{E_1 A_1 F}$ and $\Gamma_{F A_2 \dots A_c E_2}$. We now classify the possibilities with respect to the number m of intersections with the separating curve F , and to simplify matters, we will assume that $m \leq l$ – this will give a sufficiently good lower bound. This restricts the combinatorial possibilities of the intersection of γ with $\Gamma_{E_1 A_1 F}$ to one: see figure 7. The same argument as in the proof of Theorem 2.5 gives

$$(12) \quad \mathcal{N}(\mathfrak{c}, l, k, L) \geq \sum_{m \leq l} \mathcal{N}(\mathfrak{c} - 1, m, k, x)(L - x),$$

which gives the following estimate by the same argument as in the proof of Theorem 2.5.

Theorem 3.1. *The number of simple geodesics not longer than L on a surface of genus 1 with c punctures is of order L^{2c} .*

FIGURE 7. Intersection of γ with $\Gamma_{E_1 A_1 F}$

4. ARBITRARY SIGNATURE

The main difference between the lower genus situation covered in the last two sections and the higher genus case considered now is in the analysis of which fractional twists give connected curves. In the low genus case, the cyclic order of the intersections of k segments with a closed loop is always the same: $1, 2, \dots, k, k', \dots, 2', 1'$, which simplifies the analysis considerably. By contrast, in higher genus, many more permutations are possible, and it is not *a priori* obvious how to deal with them. It turns out that we can avoid dealing with the problem entirely: we only count those curves which behave in a planar fashion, and these suffice for the lower bound that we seek. It is at first surprising that we will not have thrown out the baby with the bathwater, but there is a simple heuristic explanation: since any collection of simple, pair-wise non-intersecting, and pair-wise non-isotopic curves contains at most $3g - 3$ elements, any multicurve falls naturally into (at most) $3g - 3$ subsets, the curves in which are pair-wise parallel, thus the permutation group action is closer to that of S_g than of S_k (for a k component multicurve), and thus, if we assume that every permutation is equally likely, we only lose a constant factor (in fixed genus g). The argument in the rest of this section is a direct counting argument, which appears to bear out this heuristic reasoning (which seems difficult to push through directly).

The actual argument proceeds, as before, by cutting up the surface into simpler pieces.

Consider a closed surface of genus g (the case of arbitrary signature will follow by combining the analysis below with the analysis in Section 2), and cut it along a curve E to get two pieces: one, T_E , a torus with one boundary component, the other $-T_E^{g-1}$ a surface of genus $g - 1$ with one boundary component. The intersection of the simple curve γ with E will be a collection of k segments having all of their endpoints on E .

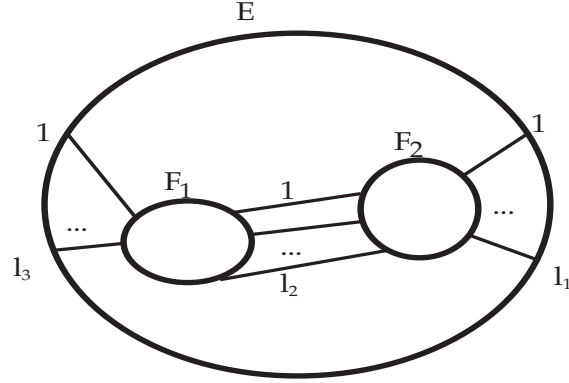


FIGURE 8. Good intersection with a punctured torus

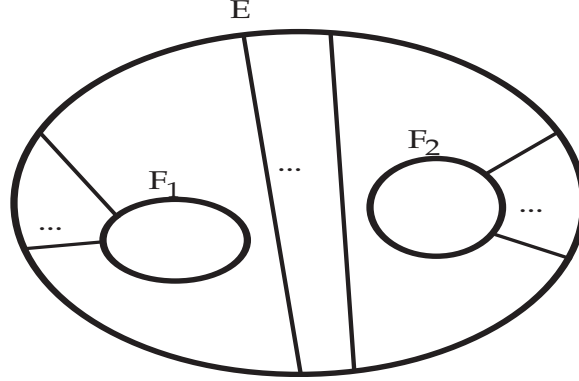


FIGURE 9. Bad intersection with a punctured torus

The piece more amenable to analysis is T_E . We analyse it by cutting it further into a thrice-punctured sphere $\Gamma_{F_1 E F_2}$. There are two combinatorial possibilities for a collection of k mutually non-intersecting segments on this surface: one is shown in figure 9, the other in figure 8.

We *forbid* the configuration shown in figure 9, since this has the wrong cycle type *vis-à-vis* E .

We have the constraints that $l_1 + l_3 = k$, while $l_1 + l_2 = l_3 + l_2$, implying that $l_1 = l_3$.

We are allowed to glue F_1 to F_2 with a twist, with the proviso that we get no closed connected components. This seems like a different sort of problem from the one we encountered in Sections 2 and 3, but luckily there is a simple trick which allows us to reduce it to that case. To wit, we glue in a disk with boundary E , and use it to connect each endpoint i to its counterpart i' . If, after further identifying F_1 to F_2 with a twist, the resulting curve is connected, obviously no circle components were created. However, this new problem is exactly the one analysed in Section 2. In particular, this tells

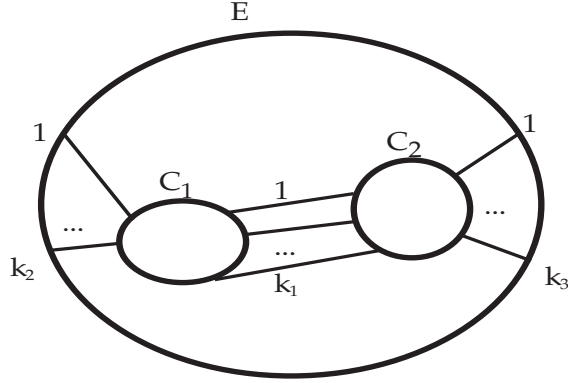


FIGURE 10. Top half of the twice-punctured torus

us that a positive proportion of the fractional twists are allowed. Now, the total length of a collection as a function of l_1 , l_3 , and the number τ of twists is bounded above by $c_1 k + c_2 l_3 + c_3 \tau$, and thus the total number of systems of length bounded above by L is at least $c(L - c_1 k)$.

To analyze the other piece, T_E^g , we use an inductive decomposition, much as before. The case of $g = 1$ was done above. If $g > 1$, we cut the surface into a torus T_{EF} with two boundary components, and a T_F^{g-1} . We are now reduced to analysing the torus T_{EF} . We want information about the collections of segments which intersect E k times, intersect F l times, and have a total of $\frac{k+l}{2}$ connected components. In order to do this we (yet again) cut T_{EF} into two thrice-punctured spheres $\Gamma_{EC_1C_2}$ and $\Gamma_{FC_1C_2}$. Let the number of intersections with C_1 be m , while the number of intersections with C_2 be n . By our requirement on the permutation type of intersection with E , the intersection of our system with $\Gamma_{EC_1C_2}$ must look like Figure 10, and similarly for $\Gamma_{FC_1C_2}$ (Figure 11). We know, furthermore, that $k_2 + k_3 = k$, $k_1 + k_2 = m$, $k_1 + k_3 = n$, which implies that

$$k_1 = \frac{m + n - k}{2}, \quad k_2 = \frac{k + m - n}{2}, \quad k_3 = \frac{k + n - m}{2}.$$

We now have to worry about two problems. One is that certain “horizontal” strands (those between C_1 and C_2) might close up into loops. The second is that certain strands might enter and leave through the boundary component F . Either way, we would get a non-connected multicurve. We deal with both of these by, first, imposing two additional inequalities. $2k_1 \leq l_1$, and $2l_1 \leq \min(m, n)$. We then allow only those twists which connect one of the k_1 strands to one of the l_1 strands (there are at least $l_1 - k_1$ such), and in addition imposing the condition that the horizontal loops do not close up – this is not extremely restrictive, since there we can twist by at least $k_1/2$ around C_1 , and also by $k_1/2$ around C_2 . (see figures 10 and for notation).

It is not hard to see that, subject to these restrictions, any strand coming into C_1 from below will either leave straightaway upwards, or will cycle

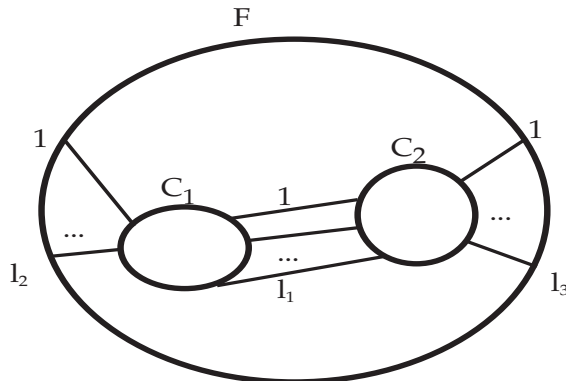


FIGURE 11. Bottom half of the twice-punctured torus

around between C_1 and C_2 for a while before leaving in that same manner. Likewise for strands coming into C_2 from below. The same sort of counting as before will show that the set of permissible multicurves of length bounded above by x is at least cx^5 , since these corresponds to points in a cone in 5 dimensions (corresponding to twisting around c_1 , twisting around c_2 , and the parameters l, m, n). Our constraints are all inequality constraints, and thus will cut out a non-degenerate cone. The rest of the inductive argument is as in Sections 2 and 3, and so for compact surfaces we obtain the claimed result

Theorem 4.1. *The number of simple geodesics of length bounded by L on a compact surface of genus g grows like L^{6g-6}*

The estimate for arbitrary signature follows as indicated in the beginning of this section.

5. CONCLUSIONS AND MUSINGS

The reader will have noted that the estimates on the *density* of connected multicurves among all multicurves become worse and worse as the genus of our surfaces increases. This is, to an extent, a reflection of reality (in fact, it is easy to see that this density decreases exponentially as a function of $g + c$). However, it is clear that the estimates one might obtain by our methods are far from optimal.

Another observation one might make is that the methods of this paper are obviously insufficient for deriving asymptotic results (extending those for the punctured torus as in [11, 12]). While there is some possibility that the method used in the low genus case might be pushed to get results of this type, for general genus this seems hopeless. However, the very existence of an asymptotic formula is in some doubt. A problem which might be tractable by the current methods is one of finding order of growth results for curves with a bounded number of self-intersections (the estimates of Birman and Series are acknowledged by the authors not to be sharp).

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MATHEMATICS DEPARTMENT, UNIVERSITY OF MANCHESTER, OXFORD ROAD, MANCHESTER

Current address: Université Paris-Sud, Orsay and Institut Henri Poincaré, Paris

E-mail address: `irivin@ma.man.ac.uk`